

On the Magnetization of a Charged Bose Gas in the Canonical Ensemble

Horia D. Cornean

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764,
70700 Bucharest, Romania;
E-mail: hcornean@imar.ro ; cornean@barutu.fizica.unibuc.ro

Received: 15 July 1999 / Accepted: 29 November 1999

Abstract: Consider a charged Bose gas without self-interactions, confined in a three dimensional cubic box of side $L \geq 1$ and subjected to a constant magnetic field $B \neq 0$. If the bulk density of particles ρ and the temperature T are fixed, then define the canonical magnetization as the partial derivative with respect to B of the reduced free energy. Our main result is that it admits thermodynamic limit for all strictly positive ρ , T and B . It is also proven that the canonical and grand canonical magnetizations (the last one at fixed average density) are equal up to the surface order corrections.

1. Introduction

Much work has been done on the thermodynamic behavior of large systems composed from independent quantum particles in the presence of external magnetic fields. As is well known, the fundamental problem consists in proving the existence of the thermodynamic limit for the potentials and the equations of state defined at finite volume. In the particular case of the canonical magnetization (defined as the partial derivative with respect to the magnetic field of the reduced free energy), one has to prove that the derivative (performed at finite volume) commutes with the thermodynamic limit of the reduced free energy. Although the quantum canonical ensemble is (from the physical point of view) the most important one, most of the previous works were carried out either using the Maxwell-Boltzmann statistics or in the framework of the quantum grand canonical ensemble, because in those settings, many physically relevant quantities can be expressed employing the integral kernel of the Gibbs semigroup associated to the one particle problem. Moreover, one is able to go beyond the bulk terms and investigate finite size effects.

Take for example the grand canonical pressure of a quantum gas in a constant magnetic field. The rigorous proof of its thermodynamic limit goes back at

least to Angelescu and Corciovei [A, A-C]; its surface correction (in the regime in which the fugacity is less than one) was obtained by Kunz [K]. As for the Maxwell-Boltzmann magnetization, nice results were obtained by Macris *et al* [M-M-P 1,2]; they wrote down even the corner corrections. Notice that in these papers the domain Λ was allowed to be more general, typically convex with piecewise smooth boundary.

Another result concerning the thermodynamic limit and the surface corrections for the magnetization and susceptibility of a Fermi gas at zero magnetic field was obtained by Angelescu *et al* [A-B-N 2]. Because this paper motivated our work, we are giving some more details about it.

Firstly, as in our setting, their domain was a rectangular parallelepiped and the magnetic field oriented after the third direction. They defined the grand canonical magnetization $m_\Lambda(\beta, z)$ (susceptibility $\chi_\Lambda(\beta, z)$) as the first (second) derivative with respect to the magnetic field of the grand canonical pressure at $B = 0$, for all $z \in \mathbb{C} \setminus (-\infty, -1]$. Their main result can be roughly stated as follows:

- i. $m_\Lambda(\beta, z) = 0, \forall z \in \mathbb{C} \setminus (-\infty, -1]$;
- ii. There exists $\chi_\infty(\beta, z)$ analytic in $\mathbb{C} \setminus (-\infty, -1]$ such that for any compact $K \subset \mathbb{C} \setminus (-\infty, -1]$ one has:

$$\lim_{\Lambda \rightarrow \infty} \sup_{z \in K} |\chi_\Lambda(\beta, z) - \chi_\infty(\beta, z)| = 0.$$

More than that, they gave even the surface correction for susceptibility and proved that this expansion is uniform on compacts. Because the relation between the fugacity and the grand canonical average density of Fermi particles can be always inverted, they were able to express the grand canonical susceptibility in terms of the canonical parameters ρ and β . Let us stress that $B = 0$ and Λ a rectangular parallelepiped were crucial ingredients in [A-B-N 2], the uniform convergence on compacts being obtained via a substantial use of the explicit formula of the integral kernel of the Gibbs semigroup associated to the Dirichlet Laplacian.

In this paper, we are studying the “true” canonical problem for a Bose gas at nonzero magnetic field $B_0 > 0$ (in order to avoid the Bose condensation). Using a standard procedure (see [K-U-Z, H]) of deriving the canonical partition function from the grand canonical pressure (see (2.27)), we are able to transform the uniform convergence on compacts of the grand canonical magnetization (see Lemma 1) into a pointwise convergence (β, ρ fixed and $L \rightarrow \infty$) of the canonical magnetization; this result is given in Theorem 2. Moreover, we obtain that the canonical magnetization m_L (see (2.29)) and the grand canonical magnetization at fixed average density (see (2.30)) are equal up to the surface order corrections.

Two natural questions arise: what about Fermi statistics and what about higher derivatives with respect to B (the susceptibility for example)? Partial answers and a few open problems are outlined at the end of the proofs.

2. Preliminaries and the Results

Let $\Lambda = \{\mathbf{x} \in \mathbb{R}^3 \mid -\frac{L}{2} < x_j < \frac{L}{2}, j \in \{1, 2, 3\}\}$, $L > 1$, be a cubic box with its side equal to L . Then the “one particle” Hilbert space is $\mathcal{H}_{1,L} := \mathbf{L}^2(\Lambda)$; denote with $\mathcal{H}_{n,L}$ the proper subspace of $\otimes_{j=1}^n \mathcal{H}_{1,L} \cong \mathbf{L}^2(\Lambda^n)$ which contains all totally

symmetric functions. Denote with $\mathcal{H}_{0,L} = \mathbb{C}$ the space with no particles; then the Fock space is defined as $\mathcal{F}_L := \bigoplus_{n \geq 0} \mathcal{H}_{n,L}$. One can introduce the “number of particles” operator N_L as the unique self-adjoint extension of the multiplication with n on each $\mathcal{H}_{n,L}$.

Assume that the particles (each having an electric charge e) are subjected to a constant magnetic field $\mathbf{B} = B\mathbf{e}_3$, which corresponds to a magnetic vector potential $B\mathbf{a} = \frac{B}{2}\mathbf{e}_3 \wedge \mathbf{x}$. If c stands for the speed of light, define $\omega := (e/c)B$. Then the “one particle” Hamiltonian (denoted with $H_{1,L}(\omega)$) will be the Friederichs extension of the symmetric and positive operator $\frac{1}{2}(-i\nabla - \omega\mathbf{a})^2$ defined on $C_0^\infty(\Lambda)$.

Due to the regularity of Λ , $H_{1,L}(\omega)$ is essentially self-adjoint on

$$D = \{f \in C^2(\Lambda) \cap C^1(\bar{\Lambda}), f|_{\partial\Lambda} = 0, \Delta f \in \mathbf{L}^2(\Lambda)\}.$$

The Hamiltonian which describes n particles reads as:

$$H_{n,L}(\omega) = \underbrace{H_{1,L}(\omega) \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes H_{1,L}(\omega)}_{\text{“n” terms}}. \quad (2.1)$$

The second quantized Hamiltonian $H_L(\omega)$ is defined as the unique self-adjoint operator on \mathcal{F}_L whose restrictions to $\mathcal{H}_{n,L}$ coincide with $H_{n,L}(\omega)$.

If $T > 0$ stands for the temperature and $\mu \in \mathbb{R}$ for the chemical potential, then define $\beta = \frac{1}{k_B T} > 0$ and $z = \exp(\beta\mu)$ (the fugacity), where k_B is the Boltzmann constant. When working in the canonical ensemble, one considers that the bulk density of particles ρ is constant, therefore the number of particles is defined as $N(L) := \rho L^3$.

As is well known (see [R-S 4]), $H_{1,L}(\omega)$ is positive, unbounded and has compact resolvent; these imply that its spectrum is purely discrete with accumulation point at infinity. Moreover, from the min-max principle it follows:

$$\inf \sigma(H_{1,L}(\omega)) \geq \inf \sigma(H_{1,\infty}(\omega)) = \frac{\omega}{2}. \quad (2.2)$$

It is also known that the semigroup $W_L(\beta, \omega) := \exp(-\beta H_{1,L}(\omega))$ is trace class and admits an integral kernel $G_{\omega,L}(\mathbf{x}, \mathbf{x}'; \beta)$, which is continuous in both its “spatial” variables. The diamagnetic inequality at finite volume (see [B-H-L]) reads as:

$$|G_{\omega,L}(\mathbf{x}, \mathbf{x}'; \beta)| \leq G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta) \leq \frac{1}{(2\pi\beta)^{3/2}} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|^2}{2\beta}\right). \quad (2.3)$$

If $\mathcal{I}_1(\mathbf{L}^2(\Lambda))$ denotes the Banach space of trace class operators, it follows that:

$$\|W_L(\beta, \omega)\|_{\mathcal{I}_1} = \text{tr } W_L(\beta, \omega) \leq \frac{L^3}{(2\pi\beta)^{3/2}}. \quad (2.4)$$

Denote with $\{E_j(\omega)\}_{j \in \mathbf{N}}$ the set of the eigenvalues of $H_{1,L}(\omega)$. If $\mu < 0$, the grand canonical partition function reads as:

$$\Xi_L(\beta, z, \omega) = \text{tr}_{\mathcal{F}_L} \exp[-\beta(H_L(\omega) - \mu N_L)] = \prod_{j=0}^{\infty} [1 - z \exp(-\beta E_j(\omega))]^{-1}. \quad (2.5)$$

The canonical partition function of our system is:

$$Z_L(\beta, \rho, \omega) = \text{tr}_{\mathcal{H}_{N(L),L}} \exp(-\beta H_{N(L),L}(\omega)). \quad (2.6)$$

The link between them is contained in the following equality:

$$\Xi_L(\beta, z, \omega) = \sum_{n=0}^{\infty} z^n \text{tr}_{\mathcal{H}_{n,L}} \exp[-\beta H_{n,L}]. \quad (2.7)$$

Throughout the entire paper, by $\log z$ we shall understand the logarithm function restricted to $\mathbb{C} \setminus (-\infty, 0]$.

Let \mathcal{C} be a contour which surrounds the origin, does not intersect the cut $[1, \infty)$ but contains the spectrum of the trace class operator zW_L , where $z \in \mathbb{C} \setminus [\exp(\beta\omega/2), \infty)$. Let $q(\xi) = \frac{1}{\xi} \log(1 - \xi)$ be an analytic function in the interior of \mathcal{C} . Define the following bounded operator:

$$q(zW_L) = \frac{1}{2\pi i} \int_{\mathcal{C}} d\xi q(\xi)(\xi - zW_L)^{-1}. \quad (2.8)$$

It is easy to see that $\log(1 - zW_L) = zW_L \cdot q(zW_L)$ and using (2.5) one obtains:

$$\log \Xi_L(\beta, z, \omega) = -\text{tr}(zW_L \cdot q(zW_L)). \quad (2.9)$$

Employing the above expression, one can easily prove that the grand canonical potential (seen as a function of z) is analytic in $\mathbb{C} \setminus [\exp(\beta\omega/2), \infty)$. When $|z| < 1$, (2.9) becomes:

$$\log \Xi_L(\beta, z, \omega) = \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(W_L^n) = \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\int_{\Lambda} d\mathbf{x} G_{\omega,L}(\mathbf{x}, \mathbf{x}; n\beta) \right). \quad (2.10)$$

The grand canonical pressure and density are defined as:

$$P_L(\beta, z, \omega) := \frac{1}{\beta L^3} \log \Xi_L(\beta, z, \omega) = -\frac{1}{\beta L^3} \sum_j \log \left(1 - z e^{-\beta E_j(\omega)} \right), \quad (2.11)$$

and

$$\rho_L(\beta, z, \omega) := \beta z \frac{\partial P_L}{\partial z}(\beta, z, \omega). \quad (2.12)$$

Let us remark that $\rho_L(\beta, x, \omega)$ is an increasing function if $0 < x < \exp(\beta\omega/2)$:

$$\frac{\partial \rho_L}{\partial x}(\beta, x, \omega) = 1/L^3 \text{tr}[(1 - xW_L)^{-2} W_L] > 0. \quad (2.13)$$

The proof of the thermodynamic limit for these two quantities goes back at least to Angelescu and Corciovei [A, A-C]. Because this result plays an important role in our work, we shall reproduce it here. In order to do that, let us define ($\omega > 0$):

$$P_{\infty}(\beta, z, \omega) := \omega \frac{1}{(2\pi\beta)^{3/2}} \sum_{k=0}^{\infty} g_{3/2} \left(z e^{-(k+1/2)\omega\beta} \right), \quad (2.14)$$

and

$$\rho_\infty(\beta, z, \omega) := \beta z \frac{\partial P_\infty}{\partial z}(\beta, z, \omega) = \beta \omega \frac{1}{(2\pi\beta)^{3/2}} \sum_{k=0}^{\infty} g_{1/2} \left(z e^{-(k+1/2)\omega\beta} \right), \quad (2.15)$$

where $g_\sigma(\zeta)$ are the usual Bose functions:

$$g_\sigma(\zeta) = \frac{\zeta}{\Gamma(\sigma)} \int_0^\infty dt \frac{t^{\sigma-1} e^{-t}}{1 - \zeta e^{-t}}, \quad (2.16)$$

analytic in $\mathbb{C} \setminus [1, \infty)$ and if $|\zeta| < 1$, they are given by the following expansion:

$$g_\sigma(\zeta) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n^\sigma}.$$

Then the following result is true (see [A, A-C]):

Theorem 1. *Let $K \subset \mathbb{C} \setminus [\exp(\beta\omega/2), \infty)$ be a compact set. Then the grand canonical pressure and density admit the thermodynamic limit i.e.:*

$$\lim_{L \rightarrow \infty} \sup_{z \in K} |P_L(\beta, z, \omega) - P_\infty(\beta, z, \omega)| = 0, \quad (2.17)$$

and

$$\lim_{L \rightarrow \infty} \sup_{z \in K} |\rho_L(\beta, z, \omega) - \rho_\infty(\beta, z, \omega)| = 0. \quad (2.18)$$

Firstly, because P_L and ρ_L are analytic functions, then via the Cauchy integral formula it follows that all their complex derivatives admit a limit which is uniform on compacts. In particular:

$$\lim_{L \rightarrow \infty} \sup_{z \in K} \left| \frac{\partial \rho_L}{\partial z}(\beta, z, \omega) - \frac{\partial \rho_\infty}{\partial z}(\beta, z, \omega) \right| = 0. \quad (2.19)$$

It can be seen from (2.15) that $\lim_{x \nearrow e^{\beta\omega/2}} \rho_\infty(\beta, x, \omega) = \infty$, which means that the Bose condensation is absent when a nonzero magnetic field is present. A very important consequence of the theorem is that the relation between the fugacity and density can be inverted for all temperatures and moreover, if $0 < x_\infty(\beta, \rho, \omega) < e^{\beta\omega/2}$ is the unique real and positive solution of the equation $\rho_\infty(\beta, x, \omega) = \rho$ and if $x_L(\beta, \rho, \omega)$ is the unique real and positive solution which solves $\rho_L(\beta, x, \omega) = \rho$, then $\lim_{L \rightarrow \infty} x_L = x_\infty$.

Let us perform the Legendre transform at finite volume:

$$\tilde{f}_L(\beta, \rho, \omega) := -P_L(\beta, x_L(\beta, \rho, \omega), \omega) + \frac{\rho}{\beta} \log x_L(\beta, \rho, \omega). \quad (2.20)$$

A straightforward result is that \tilde{f}_L has the following limit:

$$f_\infty(\beta, \rho, \omega) := -P_\infty(\beta, x_\infty(\beta, \rho, \omega), \omega) + \frac{\rho}{\beta} \log x_\infty(\beta, \rho, \omega). \quad (2.21)$$

Let us denote with $\frac{\partial W_L}{\partial \omega}(\beta, \omega_0)$ the following integral which makes sense in the norm topology of $B(\mathbf{L}^2(A))$ (see [A-B-N 1]):

$$- \int_0^\beta d\tau W_L(\beta - \tau, \omega_0) [\mathbf{a} \cdot (\mathbf{p} - \omega_0 \mathbf{a})] W_L(\tau, \omega_0). \quad (2.22)$$

A particular case of the problem treated in [A-B-N 1] is that $\frac{\partial W_L}{\partial \omega}(\beta, \omega_0)$ is even trace class and moreover, for $\delta\omega$ sufficiently small one has:

$$\left\| W_L(\beta, \omega_0 + \delta\omega) - W_L(\beta, \omega_0) - \delta\omega \frac{\partial W_L}{\partial \omega} \right\|_{\mathcal{I}_1} = \mathcal{O}((\delta\omega)^2). \quad (2.23)$$

Using for $P_L(\beta, z, \omega_0)$ the following expression:

$$-\frac{1}{\beta L^3} \operatorname{tr}[\log(1 - zW_L(\beta, \omega_0))],$$

then the estimate (2.23) justifies the definition of the grand canonical magnetization:

$$\Gamma_L(\beta, z, \omega_0) := -\frac{e}{c} \frac{\partial P_L}{\partial \omega}(\beta, z, \omega_0) = -\frac{e}{c\beta L^3} \operatorname{tr} \left[(1 - zW_L(\beta, \omega_0))^{-1} \frac{\partial W_L}{\partial \omega} \right]. \quad (2.24)$$

From its definition, one can easily see that Γ_L has the same domain of analyticity in z . Now let us define the natural candidate for its thermodynamic limit:

$$\Gamma_\infty(\beta, z, \omega_0) := -\frac{e}{c} \frac{\partial P_\infty}{\partial \omega}(\beta, z, \omega_0). \quad (2.25)$$

Our main technical result is presented in the following lemma:

Lemma 1. *Let $K \subset \mathbb{C} \setminus [\exp(\beta\omega/2), \infty)$ be a compact set. Then the grand canonical magnetization admits the thermodynamic limit i.e.:*

$$\lim_{L \rightarrow \infty} \sup_{z \in K} |\Gamma_L(\beta, z, \omega) - \Gamma_\infty(\beta, z, \omega)| = 0. \quad (2.26)$$

Let us go back to the canonical ensemble. From (2.7), (2.11) and (2.6), one can write down an useful representation of the canonical partition function:

$$Z_L(\beta, \rho, \omega) = \frac{1}{2\pi i} \int_{C_1} d\xi \frac{1}{\xi} \left[\frac{\exp\left(\frac{\beta}{\rho} P_L(\beta, \xi, \omega)\right)}{\xi} \right]^{N(L)}, \quad (2.27)$$

where C_1 is a contour which surrounds the origin and avoids the cut. The reduced free energy reads as:

$$f_L(\beta, \rho, \omega) := -\frac{1}{\beta L^3} \log Z_L(\beta, \rho, \omega). \quad (2.28)$$

The canonical magnetization is defined as follows:

$$m_L(\beta, \rho, \omega) := \frac{e}{c} \frac{\partial f_L}{\partial \omega}(\beta, \rho, \omega). \quad (2.29)$$

We expect that m_L should be close to the following quantity:

$$\tilde{m}_L(\beta, \rho, \omega) := \frac{e}{c} \frac{\partial \tilde{f}_L}{\partial \omega}(\beta, \rho, \omega) = \Gamma_L(\beta, x_L(\beta, \rho, \omega), \omega), \quad (2.30)$$

which converges to $\Gamma_\infty(\beta, x_\infty(\beta, \rho, \omega), \omega)$.

We are able now to give our main result:

Theorem 2. *Fix $0 < \delta < 1/2$. For all strictly positive temperatures, bulk densities and magnetic fields, the canonical magnetization $m_L(\beta, \rho, \omega)$ admits the thermodynamic limit. Moreover, there exist two positive constants $C_\delta(\beta, \rho, \omega)$ and $L_\delta(\beta, \rho, \omega)$ such that for all $L \geq L_\delta$ one has:*

$$|m_L - \tilde{m}_L| \leq C_\delta L^{-3/2+\delta}. \quad (2.31)$$

Remark. It is clear that \tilde{m}_L is a much more convenient quantity. If (at least for dilute gases) one would be able to write down an expansion for \tilde{m}_L which takes into account the surface corrections:

$$\tilde{m}_L = m_\infty + \frac{1}{L} m_S + o(1/L), \quad (2.32)$$

then the estimate (2.31) would imply that the same expansion is true for m_L , too.

3. The Proof of Theorem 2

At this point, we shall consider that Lemma 1 is true and give its proof in the next section. In order to simplify the notations, we shall drop the dependence on β , ρ and ω but we shall reintroduce it when needed.

The main idea of the proof consists in isolating the principal part of the integral from (2.27). Although this procedure is far from being new (in the physical literature it is known as the Darwin-Fowler method; see [K-U-Z, H] and references therein), we decided to give a rather detailed proof in order to have a clearer image of the remainder from (2.31). Firstly, let us choose the contour C_1 as follows:

$$C_1 := \{x_L e^{i\phi}, \phi \in [-\pi, \pi]\}. \quad (3.1)$$

Using (2.20), the formula (2.27) can be rewritten as:

$$\begin{aligned} Z_L &= \exp(-\beta \tilde{f}_L L^3) \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \exp \left[\frac{N(L) \beta (P_L(x_L e^{i\phi}) - P_L(x_L))}{\rho} \right] e^{-iN(L)\phi} = \\ &= \exp(-\beta \tilde{f}_L L^3) N(L)^{-1/2} \mathcal{A}_L, \end{aligned} \quad (3.2)$$

where $\mathcal{A}_L(\beta, \rho, \omega)$ is given by:

$$\mathcal{A}_L = \frac{\sqrt{N(L)}}{2\pi} \int_{-\pi}^{\pi} d\phi e^{[N(L)\beta/\rho (\Re P_L(x_L e^{i\phi}) - P_L(x_L))]} e^{iN(L)[\beta \Im P_L(x_L e^{i\phi})/\rho - \phi]}. \quad (3.3)$$

Denote with $\tilde{p}_\infty(\phi)$ the function given by $\Re P_\infty(x_\infty e^{i\phi})$, and with $\tilde{p}_L(\phi)$ the function given by $\Re P_L(x_L e^{i\phi})$.

We shall prove that $\frac{d^2 \tilde{p}_\infty}{d\phi^2}(0) < 0$ and that the following limit is true:

$$\lim_{L \rightarrow \infty} \mathcal{A}_L := \mathcal{A}_\infty = \sqrt{-\frac{\rho}{2\pi\beta \frac{d^2 \tilde{p}_\infty}{d\phi^2}(0)}}. \quad (3.4)$$

Firstly, let us remark that $\overline{P_L(\xi)} = P_L(\bar{\xi})$, where the over line means complex conjugation. Let $0 < \epsilon < \frac{e^{\beta\omega/2} - x_\infty}{2}$ be a fixed positive number, and let C_ϵ be a circle centered in x_∞ with radius ϵ . If ξ belongs to the interior of C_ϵ then the Cauchy integral formula gives ($L \leq \infty$):

$$P_L(\xi) = \frac{1}{2\pi i} \int_{C_\epsilon} d\zeta \frac{P_L(\zeta)}{\zeta - \xi}. \quad (3.5)$$

Take $0 < \delta < 1/2$ and define $\phi_L := N(L)^{-1/2+\delta/3}$. Then for L big enough and $|\phi| \leq \phi_L$, one has that $x_L e^{i\phi}$ belongs to the interior of $C_{\epsilon/2}$ and (3.5) implies:

$$\tilde{p}_L(\phi) = \frac{1}{4\pi i} \int_{C_\epsilon} d\zeta P_L(\zeta) \left(\frac{1}{\zeta - x_L e^{i\phi}} + \frac{1}{\zeta - x_L e^{-i\phi}} \right). \quad (3.6)$$

Theorem 1 implies now that:

$$\lim_{L \rightarrow \infty} \frac{d^2 \tilde{p}_L}{d\phi^2}(0) = \frac{d^2 \tilde{p}_\infty}{d\phi^2}(0) \quad \text{and} \quad \forall |\phi| \leq \phi_L, \quad \left| \frac{d^3 \tilde{p}_L}{d\phi^3}(\phi) \right| \leq \text{const}. \quad (3.7)$$

Denote with $s_L(\phi)$ the function $\Im P_L(x_L e^{i\phi})$. In a similar way, one can obtain that for all $|\phi| \leq \phi_L$:

$$\left| \frac{d^3 s_L}{d\phi^3}(\phi) \right| \leq \text{const}. \quad (3.8)$$

Finally, let us prove (3.4). This will be done in two steps:

1. The first one consists in showing that the contribution to \mathcal{A}_L coming from the region $|\phi| \geq \phi_L$ is exponentially small in L ; we shall use that $\tilde{p}_L(\phi)$ is an even function, is decreasing on the interval $[0, \pi]$ and has a non-degenerate maximum in 0;
2. The second one consists in a more careful study of the case in which $|\phi| \leq \phi_L$; the most important thing here is showing that the oscillations of the imaginary part are small.

Let us prove 1. Firstly, because $\tilde{p}_L(\phi) = \tilde{p}_L(-\phi)$, it is sufficient to look only at $\phi \in [\phi_L, \pi]$. From (2.11) one has:

$$\tilde{p}_L(\phi) = -\frac{1}{2\beta L^3} \sum_j \log \left[(1 - x_L \cos \phi e^{-\beta E_j})^2 + x_L^2 \sin^2 \phi e^{-2\beta E_j} \right], \quad (3.9)$$

and:

$$\frac{d\tilde{p}_L}{d\phi}(\phi) = -\frac{x_L \sin \phi}{\beta L^3} \sum_j \frac{e^{-\beta E_j}}{|1 - x_L e^{i\phi} e^{-\beta E_j}|^2}. \quad (3.10)$$

This means that \tilde{p}_L is decreasing on $[0, \pi]$. Its second derivative in 0 reads as:

$$\frac{d^2 \tilde{p}_L}{d\phi^2}(0) = -\frac{x_L}{\beta L^3} \sum_j \frac{e^{-\beta E_j}}{|1 - x_L e^{-\beta E_j}|^2}. \quad (3.11)$$

For L big enough, one has that $x_L \geq x_\infty/2$. The above equality implies:

$$\frac{d^2 \tilde{p}_L}{d\phi^2}(0) \leq -\frac{x_\infty}{2\beta} \frac{1}{L^3} \text{tr} W_L. \quad (3.12)$$

To obtain an uniform estimate in L , we employ (see [K]):

$$\frac{1}{L^3} \text{tr} W_L = \frac{1}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)} (1 + \mathcal{O}(1/L)). \quad (3.13)$$

Then (3.7), (3.12) and (3.13) imply that for L sufficiently large:

$$\frac{d^2 \tilde{p}_L}{d\phi^2}(0) \leq -\frac{x_\infty}{4\beta} \frac{1}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)} \quad \text{and} \quad \frac{d^2 \tilde{p}_\infty}{d\phi^2}(0) < 0. \quad (3.14)$$

From (3.14) and (3.7) it follows that for all $\phi \in [\phi_L, \pi]$ one has:

$$\tilde{p}_L(\phi) - P_L(x_L) \leq \tilde{p}_L(\phi_L) - \tilde{p}_L(0) \leq \frac{1}{4} \frac{d^2 \tilde{p}_\infty}{d\phi^2}(0) \phi_L^2, \quad (3.15)$$

or in other words,

$$\exp \left\{ \frac{N(L)\beta}{\rho} [\tilde{p}_L(\phi) - P_L(x_L)] \right\} \leq \exp \left[\frac{\beta}{4\rho} \frac{d^2 \tilde{p}_\infty}{d\phi^2}(0) N(L)^{2\delta/3} \right], \quad (3.16)$$

therefore the contribution to \mathcal{A}_L coming from $|\phi| \geq \phi_L$ is exponentially small.

Now let us study the region $|\phi| \leq \phi_L$. We are mainly interested in the behavior of the imaginary part of the pressure (see (3.8)). Similarly as in (3.9), one has:

$$s_L(\phi) = -\frac{1}{\beta L^3} \sum_j \arg(1 - x_L e^{i\phi} e^{-\beta E_j}) = \frac{1}{\beta L^3} \sum_j \arctan \frac{x_L \sin \phi e^{-\beta E_j}}{1 - x_L \cos \phi e^{-\beta E_j}}. \quad (3.17)$$

By direct computation, one obtains:

$$s_L(0) = 0, \quad \frac{ds_L}{d\phi}(0) = \frac{\rho_L(x_L)}{\beta} = \frac{\rho}{\beta} \quad \text{and} \quad \frac{d^2 s_L}{d\phi^2}(0) = 0. \quad (3.18)$$

Together with (3.8), one has:

$$\frac{\beta}{\rho} s_L(\phi) - \phi = \mathcal{O}(\phi_L^3). \quad (3.19)$$

We are interested now in the following integral:

$$\frac{\sqrt{N(L)}}{2\pi} \int_{-\phi_L}^{\phi_L} d\phi e^{[N(L)\beta/\rho(\Re P_L(x_L e^{i\phi}) - P_L(x_L))]} e^{iN(L)[\beta \Im P_L(x_L e^{i\phi})/\rho - \phi]}. \quad (3.20)$$

Changing the variable in $t = \sqrt{N(L)}\phi$ one obtains:

$$\frac{1}{2\pi} \int_{-N(L)^{\delta/3}}^{N(L)^{\delta/3}} dt e^{\frac{\beta}{2\rho} \frac{d^2 \tilde{P}_L}{d\phi^2}(0)} t^2 e^{\mathcal{O}[N(L)^{-1/2+\delta}]}. \quad (3.21)$$

Using (3.14), (3.7) and the Lebesgue dominated convergence theorem, it is easy to see that the above integral converges to (3.4).

From (2.28) and (3.2), the reduced free energy reads as:

$$f_L = \tilde{f}_L + \frac{\log N(L)}{2\beta L^3} - \frac{\log \mathcal{A}_L}{\beta L^3}. \quad (3.22)$$

From (2.29) and (2.30) it follows:

$$m_L = \tilde{m}_L - \frac{e}{\beta c L^3} \frac{1}{\mathcal{A}_L} \frac{\partial \mathcal{A}_L}{\partial \omega}, \quad (3.23)$$

or:

$$\begin{aligned} m_L - \tilde{m}_L &= \\ &= \frac{N(L)^{1/2}}{2\pi \mathcal{A}_L} \int_{-\pi}^{\pi} d\phi e^{[\frac{N(L)\beta}{\rho}(P_L(x_L e^{i\phi}) - P_L(x_L))]} \left[-\frac{e}{c} \frac{d}{d\omega} (P_L(x_L e^{i\phi}) - P_L(x_L)) \right], \end{aligned} \quad (3.24)$$

where:

$$\begin{aligned} &-\frac{e}{c} \frac{d}{d\omega} (P_L(x_L e^{i\phi}) - P_L(x_L)) = \\ &= \Gamma_L(x_L e^{i\phi}) - \Gamma_L(x_L) - \frac{e}{c} \left[\frac{\partial P_L}{\partial z}(x_L e^{i\phi}) \frac{\partial x_L}{\partial \omega} e^{i\phi} - \frac{\partial P_L}{\partial z}(x_L) \frac{\partial x_L}{\partial \omega} \right]. \end{aligned} \quad (3.25)$$

From (2.12) it follows that $-\frac{e}{c} \frac{\partial \rho_L}{\partial \omega}(\beta, z, \omega) = \beta z \frac{\partial \Gamma_L}{\partial z}(\beta, z, \omega)$. This implies:

$$-\frac{e}{c} \frac{\partial x_L}{\partial \omega} = -\beta x_L \frac{\partial \Gamma_L}{\partial z}(x_L) \left[\frac{\partial \rho_L}{\partial z}(x_L) \right]^{-1}.$$

Lemma 1 and Theorem 1 imply (via the Cauchy integral formula) that the above quantity remains bounded when L goes to infinity. Performing a similar analysis of the integral from (3.24) as that one made for \mathcal{A}_L , it follows that the contribution coming from the region $|\phi| \geq \phi_L$ is exponentially small in L . When we are analyzing the region $|\phi| \leq \phi_L$, the factor $N(L)^{1/2}$ disappears when we are changing the variable, therefore $m_L - \tilde{m}_L$ will behave as the quantity from (3.25), and this one behaves like $\phi_L \sim L^{-3/2+\delta}$. The proof of (2.31) is now completed.

4. The Proof of Lemma 1

The idea of the proof is borrowed from [A]. Namely, the uniform convergence on the compacts which belong to $\mathcal{D} := \mathbb{C} \setminus [e^{\beta\omega_0/2}, \infty)$ will be obtained applying the Vitali Theorem to the sequence of analytical functions $\Gamma_L(\beta, z, \omega_0)$ when L is going to infinity. Therefore, we have to make the following three steps:

- I. Prove the uniform boundedness in L of the functions $\Gamma_L(\beta, z, \omega_0)$ on any compact $K \subset \mathcal{D}$;
- II. Identify the limit ($\Gamma_\infty(\beta, z, \omega_0)$ in our case) and prove that it has the same domain of analyticity \mathcal{D} ;
- III. Prove the existence of a set $\mathcal{D}_0 \subset \mathcal{D}$ having at least one point of accumulation, such that $\Gamma_L(\beta, z, \omega_0)$ has a pointwise convergence to $\Gamma_\infty(\beta, z, \omega_0)$ on \mathcal{D}_0 .

4.1. The proof of I. Here is the “nontrivial part” of our paper. In order to see where the main difficulty is, let us take a look at (2.24). One could try a bound on $\Gamma_L(\beta, z, \omega)$ using the following inequality (see also (2.22)):

$$\begin{aligned} \sup_{z \in K} \left| \operatorname{tr} \left[(1 - zW_L)^{-1} \frac{\partial W_L}{\partial \omega} \right] \right| &\leq \left\| \frac{\partial W_L}{\partial \omega} \right\|_{\mathcal{I}_1} \sup_{z \in K} \| (1 - zW_L)^{-1} \|_{B(\mathbf{L}^2)} \\ &\leq C(\beta, K, \omega) \left\| \frac{\partial W_L}{\partial \omega} \right\|_{\mathcal{I}_1}. \end{aligned}$$

Unfortunately, the linear growth of the vector potential \mathbf{a} leads to a bad trace norm estimate for $\frac{\partial W_L}{\partial \omega}$, which behaves like L^4 and not like L^3 as needed. In the case of the pressure [A-C], this kind of problem does not appear (see (2.9), (2.8) and (2.4)), because for the Gibbs semigroup the trace and the trace norm are equal and grow like L^3 .

Because the proof is rather lengthy, we shall outline in a few words our strategy. Firstly, having (2.8) in mind, let us define $(\omega_0, \beta, \tau > 0)$:

$$g_{\omega_0}(\xi, z; \beta, \tau) := [\xi - zW_L(\beta, \omega_0)]^{-1} zW_L(\tau, \omega_0), \quad (4.1)$$

where ξ belongs to the contour \mathcal{C} . Then the pressure will admit the following representation (see (2.9)):

$$P_L(\beta, z, \omega_0) = -\frac{1}{2\pi i} \int_{\mathcal{C}} d\xi \, q(\xi) \frac{1}{L^3} \operatorname{tr}[g_{\omega_0}(\xi, z; \beta, \beta)], \quad (4.2)$$

where the norm convergent integral from (2.8) commutes with the trace from (2.9). Let us argue that we can choose the same contour of integration if ω varies in a small interval $\Omega := [\omega_0, \omega_1]$, $0 < \omega_0 < \omega_1$. In other words, we search for a \mathcal{C} included in $\mathbb{C} \setminus [1, \infty)$ such that for all $L > 1$, $z \in K$, $\omega \in \Omega$ and $\xi \in \mathcal{C}$ one has the following bound:

$$\|[\xi - zW_L(\beta, \omega)]^{-1}\| \leq M < \infty. \quad (4.3)$$

Assume that every $z \in K$ verifies the following condition:

$$\operatorname{dist}\{z, [e^{\frac{\beta\omega_0}{2}}, \infty)\} \geq \delta > 0, \quad |z| \leq d < \infty. \quad (4.4)$$

We know that the spectrum of $W_L(\beta, \omega)$ is included in $[0, e^{-\frac{\beta\omega_0}{2}}]$ for all $L > 1$ and $\omega \in \Omega$. We claim that \mathcal{C} can be chosen as the union $\mathcal{C}_1 \cup \mathcal{C}_2$, where \mathcal{C}_2 is given by ($\eta > 0$):

$$\{(1+t, \pm\eta) \mid -\eta \leq t \leq 2d\} \cup \{(1-\eta, t) \mid -\eta \leq t \leq \eta\},$$

and \mathcal{C}_1 is chosen such that if $\xi \in \mathcal{C}_1$, then $|\xi| \geq 2d+1$. It is not difficult to prove that by choosing η sufficiently small, then:

$$\sup_{z \in K} \sup_{\xi \in \mathcal{C}} \sup_{0 \leq r \leq e^{-\frac{\beta\omega_0}{2}}} |(\xi - zr)^{-1}| \leq M < \infty, \quad (4.5)$$

and via the Spectral Theorem, (4.3) takes place.

If we manage to prove the existence of a numerical constant $c(\beta, K, \omega_0)$ such that for all $\delta\omega \in (0, \omega_1 - \omega_0)$ and $z \in K$ to have:

$$\sup_z \sup_{\delta\omega} \frac{1}{\delta\omega} |P_L(\beta, z, \omega_0 + \delta\omega) - P_L(\beta, z, \omega_0)| \leq c(\beta, K, \omega_0), \quad (4.6)$$

then the magnetization will be bounded by the same constant. This estimate is straightforward if a stronger one takes place:

$$\sup_z \sup_{\xi} \sup_{\delta\omega} \frac{1}{L^3 \delta\omega} |\text{tr}[g_{\omega_0+\delta\omega}(\xi, z; \beta, \beta)] - \text{tr}[g_{\omega_0}(\xi, z; \beta, \beta)]| \leq C(\beta, K, \omega_0). \quad (4.7)$$

Our main task will consist in constructing a trace class operator $A_{\omega_0+\delta\omega}(\xi, z; \beta)$ having the following two properties: $\text{tr} A_{\omega_0+\delta\omega}(\xi, z; \beta) = \text{tr} g_{\omega_0}(\xi, z; \beta, \beta)$ (i.e. its trace is *not* depending on $\delta\omega$) and moreover,

$$\sup_z \sup_{\xi} \sup_{\delta\omega} \frac{1}{L^3 \delta\omega} \|g_{\omega_0+\delta\omega}(\xi, z; \beta, \beta) - A_{\omega_0+\delta\omega}(\xi, z; \beta)\|_{\mathcal{I}_1} \leq C'(\beta, K, \omega_0), \quad (4.8)$$

which would clearly end the problem.

We will see that $(\omega = \omega_0 + \delta\omega)$ $A_{\omega}(\xi, z; \beta)$ can be chosen as a product $\tilde{g}_{\omega}(\xi, z; \beta, \beta/2) S_{\omega}(\beta/2)$ where the first term is bounded, the second one is trace class and moreover:

$$\sup_z \sup_{\xi} \|g_{\omega}(\xi, z; \beta, \beta/2) - \tilde{g}_{\omega}(\xi, z; \beta, \beta/2)\|_{B(\mathbb{L}^2)} \leq C_1(\beta, K, \omega_0) \delta\omega, \quad (4.9)$$

and

$$\|W_L(\beta/2, \omega) - S_{\omega}(\beta/2)\|_{\mathcal{I}_1} \leq C_2(\beta, \omega_0) L^3 \delta\omega. \quad (4.10)$$

In particular, (4.9) and (4.10) imply:

$$\|S_{\omega}(\beta/2)\|_{\mathcal{I}_1} \leq C(\beta, \omega_0) L^3 \text{ and } \sup_z \sup_{\xi} \|\tilde{g}_{\omega}(\xi, z; \beta, \beta/2)\|_{B(\mathbb{L}^2)} \leq C_3(\beta, K, \omega_0). \quad (4.11)$$

Employing these estimates together with the following identity (see (4.1)):

$$g_{\omega}(\xi, z; \beta, \beta) = g_{\omega}(\xi, z; \beta, \beta/2) W_L(\beta/2, \omega),$$

the proof of (4.8) follows easily.

The rest of this subsection is dedicated to the rigorous proofs of these estimates and will be structured in a sequence of technical propositions. We start with a well known result, given without any other comments:

Proposition 1. *The Dirichlet Laplacian defined in Λ admits a trace class semi-group which has an integral kernel given by the following formula:*

$$G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta) = \prod_{j=1}^3 g_{0,L}(x_j, x'_j; \beta), \quad (4.12)$$

where the “one dimensional” kernels read as:

$$\begin{aligned} g_{0,L}(x, x'; \beta) &= \\ &= \frac{1}{(2\pi\beta)^{1/2}} \sum_{m \in \mathbf{Z}} \left\{ \exp \left[-\frac{(x - x' + 2mL)^2}{2\beta} \right] - \exp \left[-\frac{(x + x' - 2mL - L)^2}{2\beta} \right] \right\} \\ &:= g_{0,\infty}(x, x'; \beta) + \zeta_{0,L}(x, x'; \beta). \end{aligned} \quad (4.13)$$

Using the previous proposition, one can write:

$$G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta) = G_{0,\infty}(\mathbf{x}, \mathbf{x}'; \beta) + Z_{0,L}(\mathbf{x}, \mathbf{x}'; \beta), \quad (4.14)$$

where:

$$G_{0,\infty}(\mathbf{x}, \mathbf{x}'; \beta) = \frac{1}{(2\pi\beta)^{3/2}} \exp \left(-\frac{|\mathbf{x} - \mathbf{x}'|^2}{2\beta} \right). \quad (4.15)$$

The purpose of the next proposition is to give a few properties of smoothness and localization of the reminder $Z_{0,L}(\mathbf{x}, \mathbf{x}'; \beta)$:

Proposition 2. *For all $\beta > 0$ and $L > 1$, there exist two positive numerical constants c_1 and c_2 such that:*

- i. $\left| \frac{\partial Z_{0,L}}{\partial x_j} \right| (\mathbf{x}, \mathbf{x}'; \beta) \leq c_1 \frac{1+\beta}{\beta^{1/2}} G_{0,\infty}(\mathbf{x}, \mathbf{x}'; c_2\beta);$
- ii. $\max \left\{ \left| \frac{\partial Z_{0,L}}{\partial \beta} \right| (\mathbf{x}, \mathbf{x}'; \beta), \left| \frac{\partial^2 Z_{0,L}}{\partial x_j \partial x'_k} \right| (\mathbf{x}, \mathbf{x}'; \beta) \right\} \leq c_1 \frac{1+\beta}{\beta} G_{0,\infty}(\mathbf{x}, \mathbf{x}'; c_2\beta);$
- iii. $|Z_{0,L}|(\mathbf{x}, \mathbf{x}'; \beta) \leq c_1 (1+\beta) G_{0,\infty}(\mathbf{x}, \mathbf{x}'; c_2\beta).$

Proof. For $x, x' \in (-L/2, L/2)$ define:

$$\begin{aligned} \zeta_1(x, x'; \beta) &= \frac{1}{\sqrt{2\pi\beta}} \sum_{m \in \mathbf{Z} \setminus \{0\}} \exp \left(-\frac{(x - x' + 2mL)^2}{2\beta} \right), \\ \zeta_2(x, x'; \beta) &= \frac{1}{\sqrt{2\pi\beta}} \sum_{m \in \mathbf{Z}} \exp \left(-\frac{[x - x' - (2m+1)L]^2}{2\beta} \right). \end{aligned} \quad (4.16)$$

It is clear that if one obtains uniform estimates in L and β for these two quantities, the same would be true for $Z_{0,L}$, too. A very useful estimate is the following:

$$\forall t \geq 0, \quad te^{-t} = 2(t/2)e^{-t/2}e^{-t/2} \leq 2e^{-t/2}. \quad (4.17)$$

Then we have the inequalities:

$$\left| \frac{\partial \zeta_1}{\partial x} \right| (x, x'; \beta) \leq \frac{\text{const.}}{\beta} \sum_{m \neq 0} \exp \left(-\frac{(x - x' + 2mL)^2}{4\beta} \right), \quad (4.18)$$

$$\left| \frac{\partial^2 \zeta_1}{\partial x \partial x'} \right| (x, x'; \beta) \leq \frac{\text{const.}}{\beta^{3/2}} \sum_{m \neq 0} \exp \left(-\frac{(x - x' + 2mL)^2}{4\beta} \right),$$

and similarly for ζ_2 . At this point we have to control the summation over m . Let us prove the following inequality:

$$\sum_{m \neq 0} \exp \left(-\frac{(x - x' + 2mL)^2}{4\beta} \right) \leq c_1(1 + \beta) \exp \left(-\frac{(x - x')^2}{c_2\beta} \right). \quad (4.19)$$

Because $|x - x'| < L$ one has $(|m|, L \geq 1)$:

$$(x - x' + 2mL)^2 = (x - x')^2 + 4mL(x - x') + 4m^2L^2 \geq (x - x')^2 + 4(|m| - 1)L^2. \quad (4.20)$$

Therefore:

$$\begin{aligned} & \sum_{m \neq 0} \exp \left(-\frac{(x - x' + 2mL)^2}{4\beta} \right) \leq \\ & \leq \exp \left(-\frac{(x - x')^2}{4\beta} \right) 2 \left[1 + \sum_{m \geq 1} \exp \left(-\frac{1}{\beta} m \right) \right] = \\ & = 2 \exp \left(-\frac{(x - x')^2}{4\beta} \right) \left[1 + \frac{1}{2} \frac{\frac{1}{2\beta}}{\sinh \left(\frac{1}{2\beta} \right)} 2\beta \exp \left(-\frac{1}{2\beta} \right) \right] \leq \\ & \leq 2(1 + \beta) \exp \left(-\frac{(x - x')^2}{4\beta} \right). \end{aligned} \quad (4.21)$$

In order to control ζ_2 , one has to study the following quantity:

$$A := \sum_{m \in \mathbf{Z}} \exp \left(-\frac{[x + x' - (2m + 1)L]^2}{4\beta} \right). \quad (4.22)$$

Denote with $\xi = x + L/2$ and $\xi' = x' + L/2$; then $0 < \xi, \xi' < L$, $x - x' = \xi - \xi'$ and $(\xi + \xi')^2 \geq (\xi - \xi')^2$. It follows:

$$\begin{aligned} [x + x' - (2m + 1)L]^2 &= [\xi + \xi' - 2(m + 1)L]^2 \\ &= (\xi + \xi')^2 - 4(m + 1)L(\xi + \xi') + 4(m + 1)^2L^2. \end{aligned} \quad (4.23)$$

If $m \leq -1$, then:

$$[x + x' - (2m + 1)L]^2 \geq (x - x')^2 + 4(|m| - 1)L^2.$$

If $m = 0$, then:

$$[x + x' - (2m + 1)L]^2 = [(L/2 - x) + (L/2 - x')]^2 \geq (x - x')^2.$$

If $m \geq 1$, then:

$$[x + x' - (2m + 1)L]^2 \geq (\xi + \xi')^2 + 4L^2(m + 1)(m - 1) \geq (x - x')^2 + 4(m - 1),$$

and we can repeat the summation procedure used in (4.19). Putting all these things together, the proof is completed.

The next proposition is a variant of the perturbation theory for self-adjoint Gibbs semigroups (see [H-P]). Instead of starting with a perturbation of its generator, we start with an approximation of the semigroup. Although simple, this proposition contains the main technical core of our paper.

Proposition 3. *Let $\mathcal{H} := \mathbf{L}^2(\Lambda)$ and let H be a self-adjoint and positive operator having the domain D . Fix $\beta_0 > 0$. Assume that there exists an application $0 < \beta \leq \beta_0 \rightarrow S(\beta) \in B(\mathcal{H})$ with the following properties:*

- A. $\sup_{0 < \beta \leq \beta_0} \|S(\beta)\| \leq c_1 < \infty$;
- B. *It is strongly differentiable, $\text{Ran} S(\beta) \subset D$ and $s - \lim_{\beta \searrow 0} S(\beta) = 1$;*
- C. *There exists a normly continuous application $0 < \beta \leq \beta_0 \rightarrow R(\beta) \in B(\mathcal{H})$ such that $\|R(\beta)\| \leq c_2/\beta^\alpha$ where $0 \leq \alpha < 1$ and:*

$$\frac{\partial S}{\partial \beta} f + HS(\beta)f = R(\beta)f. \quad (4.24)$$

Then the following two statements are true:

- i. *The sequence of bounded operators ($n > [1/\beta]$):*

$$T_n(\beta) := \int_{1/n}^{\beta-1/n} d\tau \exp[-(\beta - \tau)H] R(\tau)$$

converges in norm; let $T(\beta)$ be its limit;

- ii. *The following equality takes place in $B(\mathcal{H})$:*

$$\exp(-\beta H) = S(\beta) - T(\beta). \quad (4.25)$$

Proof. i. The norm convergence is assured by the integrability condition imposed on the norm of $R(\beta)$. Moreover, when β is near zero:

$$\sup_n \|T_n(\beta)\| \leq c(\alpha)\beta^{1-\alpha}, \quad (4.26)$$

therefore the same thing is true for $T(\beta)$.

- ii. Let $0 < \beta_1 < \beta < \beta_0$. If $n > [1/\beta_1]$ and $\phi \in \mathcal{H}$, define the vector:

$$\psi_n(\beta) := \exp(-\beta H)\phi - S(\beta)\phi + T_n(\beta)\phi. \quad (4.27)$$

From (4.26) and condition A it follows:

$$\lim \psi_n(\beta) := \psi(\beta) = \exp(-\beta H)\phi - S(\beta)\phi + T(\beta)\phi \text{ and } \sup_{n, \beta} \|\psi_n(\beta)\| \leq \text{const.} \quad (4.28)$$

Define $f_n(\beta) = \|\psi_n(\beta)\|^2$ and $f(\beta) = \|\psi(\beta)\|^2$. From the strong convergence to one of $S(\beta)$ when β goes to zero and from the norm convergence to zero of $T(\beta)$, it follows that $\lim_{\beta \searrow 0} f(\beta) = 0$. If we manage to prove that $f(\beta)$ is decreasing, then it would be identically zero and this would end the proof.

Notice that $T_n(\beta)$ is normly differentiable and:

$$\frac{\partial T_n}{\partial \beta} \phi + H T_n(\beta) \phi = \exp\left(-\frac{1}{n}H\right) R(\beta - 1/n) \phi. \quad (4.29)$$

The positivity of H implies:

$$\frac{\partial f_n}{\partial \beta} \leq 2 \Re[\langle \psi_n(\beta), \exp\left(-\frac{1}{n}H\right) R(\beta - 1/n) \phi - R(\beta) \phi \rangle]. \quad (4.30)$$

Using $f_n(\beta) = f_n(\beta_1) + \int_{\beta_1}^{\beta} d\tau \frac{\partial f_n}{\partial \tau}$, one has:

$$0 \leq f_n(\beta) \leq f_n(\beta_1) + \text{const} \int_{\beta_1}^{\beta} d\tau \|\exp\left(-\frac{1}{n}H\right) R(\tau - 1/n) \phi - R(\tau) \phi\|. \quad (4.31)$$

Because the above integrand is bounded on the domain of integration and has pointwise convergence to zero, the dominated convergence theorem implies that the whole integral converges to zero. Taking the limit, it follows that $f(\beta)$ is decreasing and therefore is identically zero.

Define the “magnetic phase”:

$$\varphi(\mathbf{x}, \mathbf{x}') = \mathbf{x} \cdot \mathbf{a}(\mathbf{x}') = \frac{1}{2} \mathbf{e}_3 \cdot (\mathbf{x}' \wedge \mathbf{x}). \quad (4.32)$$

We shall use this quantity as a local gauge transformation (see [C-N] for another use of this idea); namely, it will alter the magnetic vector potential by making it depend on $\mathbf{x} - \mathbf{x}'$, only: $\mathbf{a}(\mathbf{x}) - \mathbf{a}(\mathbf{x} - \mathbf{x}') = \nabla_{\mathbf{x}} \varphi(\mathbf{x}, \mathbf{x}')$. To see how this transformation acts (at a formal level) on $H_{1,L}(\omega)$, let us notice the following equation:

$$[-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(\mathbf{x})] e^{i\omega \varphi(\mathbf{x}, \mathbf{x}')} = e^{i\omega \varphi(\mathbf{x}, \mathbf{x}')} [-i\nabla_{\mathbf{x}} - \omega \mathbf{a}(\mathbf{x} - \mathbf{x}')]. \quad (4.33)$$

Proposition 4. *The bounded operator $S(\beta) \in B(\mathbf{L}^2(\Lambda))$ given by the integral kernel $e^{i\omega \varphi(\mathbf{x}, \mathbf{x}') G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta)}$, verifies the hypotheses of Proposition 3, having the following properties:*

- i. $s - \lim_{\beta \searrow 0} S(\beta) = \mathbf{1}$;
- ii. *The application $(0, \infty) \ni \beta \mapsto S(\beta)$ is strongly differentiable and uniformly bounded;*
- iii. $\text{Ran} S(\beta) \in \text{Dom}(H_{1,L}(\omega))$ and $\frac{\partial S}{\partial \beta}(\beta) f + H_{1,L}(\omega) S(\beta) f = R(\beta) f$, where $R(\beta)$ has an integral kernel $R(\mathbf{x}, \mathbf{x}'; \beta)$ given by:

$$e^{i\omega \varphi(\mathbf{x}, \mathbf{x}')} [\omega^2 \mathbf{a}^2(\mathbf{x} - \mathbf{x}') G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta) + 2i\omega \mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot \nabla_{\mathbf{x}} Z_{0,L}(\mathbf{x}, \mathbf{x}'; \beta)].$$

Proof. i. Throughout the whole section, we shall use a few well known results, which are given without proof. Let us begin with an useful boundedness criterion for integral operators (see [S]):

Let A be an integral operator, given by a continuous integral kernel $A(\mathbf{x}, \mathbf{x}') \in C(\overline{\Lambda} \times \overline{\Lambda})$. If the next inequality holds:

$$\max \left[\sup_{\mathbf{x} \in \Lambda} \int_{\Lambda} d\mathbf{x}' |A(\mathbf{x}, \mathbf{x}')|, \sup_{\mathbf{x}' \in \Lambda} \int_{\Lambda} d\mathbf{x} |A(\mathbf{x}, \mathbf{x}')| \right] \leq C < \infty, \quad (4.34)$$

then the operator norm of A in $B(\mathbf{L}^2(\Lambda))$ is bounded by C .

If c , β and $\tau < \beta$ are three strictly positive numbers, then we have the following three identities:

$$\int_{\mathbb{R}^3} d\mathbf{y} \exp \left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{c(\beta - \tau)} - \frac{|\mathbf{y} - \mathbf{x}'|^2}{c\tau} \right) = (\pi c)^{\frac{3}{2}} \left[\frac{(\beta - \tau)\tau}{\beta} \right]^{\frac{3}{2}} \exp \left(-\frac{|\mathbf{x} - \mathbf{x}'|^2}{c\beta} \right), \quad (4.35)$$

$$\int_{\mathbb{R}^3} d\mathbf{y} \frac{1}{(\pi c \beta)^{\frac{3}{2}}} \exp \left(-\frac{|\mathbf{x} - \mathbf{y}|^2}{c\beta} \right) = 1, \quad (4.36)$$

$$\left(\int_{\mathbb{R}^3} d\mathbf{y} |G_{0,\infty}(\mathbf{x}, \mathbf{y}; \beta)|^2 \right)^{\frac{1}{2}} = (2\pi\beta)^{-\frac{3}{4}}. \quad (4.37)$$

A very useful inequality will be the next one ($t > 0$, $n \geq 1$):

$$t^n \exp \left(-\frac{t^2}{\beta} \right) \leq \text{const}(n) \beta^{\frac{n}{2}} \exp \left(-\frac{t^2}{2\beta} \right). \quad (4.38)$$

Let us get back to the proof of i. Firstly, because $|e^{i\omega\varphi(\mathbf{x}, \mathbf{x}') G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta)| \leq G_{0,\infty}(\mathbf{x}, \mathbf{x}'; \beta)$, it follows that $S(\beta)$ obeys the condition (4.34) with $C \leq 1$, which means that is uniformly bounded in $\beta > 0$.

Let us show now that the operator given by the integral kernel $(e^{i\omega\varphi(\mathbf{x}, \mathbf{x}')} - 1)G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta)$ converges in norm to zero. Let us remark first that $|\varphi(\mathbf{x}, \mathbf{x}')| \leq L|\mathbf{x} - \mathbf{x}'|$ and moreover:

$$|(e^{i\omega\varphi(\mathbf{x}, \mathbf{x}')} - 1)| \leq \omega|\varphi(\mathbf{x}, \mathbf{x}')| \leq \omega L|\mathbf{x} - \mathbf{x}'|. \quad (4.39)$$

Then (using (4.38) with $n = 1$):

$$|(e^{i\omega\varphi(\mathbf{x}, \mathbf{x}')} - 1)G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta)| \leq \text{const } L \beta^{1/2} G_{0,\infty}(\mathbf{x}, \mathbf{x}'; 2\beta), \quad (4.40)$$

therefore its operator norm behaves in zero like $\beta^{1/2}$ at least. The proof of i is now straightforward.

ii. We will prove that the application is in fact normly differentiable. For $\beta > 0$ and $\delta\beta$ sufficiently small, one has:

$$\begin{aligned} S(\beta + \delta\beta)f - S(\beta)f &= \delta\beta \int_{\Lambda} d\mathbf{x}' e^{i\omega\varphi(\cdot, \mathbf{x}')} \frac{\partial G_{0,L}}{\partial \beta}(\cdot, \mathbf{x}'; \beta) f(\mathbf{x}') + \\ &+ \frac{(\delta\beta)^2}{2} \int_{\Lambda} d\mathbf{x}' e^{i\omega\varphi(\cdot, \mathbf{x}')} \frac{\partial^2 G_{0,L}}{\partial \beta^2}(\cdot, \mathbf{x}'; \tilde{\beta}) f(\mathbf{x}'), \end{aligned} \quad (4.41)$$

where $\tilde{\beta}$ is situated between β and $\beta + \delta\beta$. It is not difficult now to see that the “operator derivative” is an integral operator whose kernel is the derivative with respect to β of the initial one.

iii. Denote with D_0 the common domain of essentially self-adjointness for $H_{1,L}(\omega)$, $\omega \geq 0$:

$$D_0 = \{\psi \in C^2(\Lambda) \cap C^1(\overline{\Lambda}) \mid \psi|_{\partial\Lambda} = 0, \Delta\psi \in \mathbf{L}^2(\Lambda)\}. \quad (4.42)$$

The action of $H_{1,L}(\omega)$ on a function from D_0 is as follows:

$$[H_{1,L}(\omega)\psi](\mathbf{x}) = -(\Delta\psi)(\mathbf{x}) + 2\omega\mathbf{a}(\mathbf{x}) \cdot (\nabla\psi)(\mathbf{x}) + \omega^2\mathbf{a}^2(\mathbf{x})\psi(\mathbf{x}). \quad (4.43)$$

Now take $f \in C_0^\infty(\Lambda)$. After integration by parts, using (4.33) and the fact that $G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta)$ solves the heat equation in the interior of Λ , one obtains:

$$\begin{aligned} \langle H_{1,L}(\omega)\psi, S(\beta)f \rangle &= \int_{\Lambda^2} d\mathbf{x}' d\mathbf{x} \overline{\psi(\mathbf{x})} f(\mathbf{x}') e^{i\omega\varphi(\mathbf{x}, \mathbf{x}')} \cdot \\ &\cdot [-\Delta_{\mathbf{x}} + 2\omega\mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot \nabla_{\mathbf{x}} + \omega^2\mathbf{a}^2(\mathbf{x} - \mathbf{x}')] G_{0,L}(\mathbf{x}, \mathbf{x}'; \beta) \\ &= -\langle \psi, S'(\beta)f \rangle + \langle \psi, R(\beta)f \rangle. \end{aligned} \quad (4.44)$$

The result follows easily after a density argument and with the remark that:

$$\mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot \nabla_{\mathbf{x}} G_{0,\infty}(\mathbf{x}, \mathbf{x}'; \beta) = 0. \quad (4.45)$$

Finally, let us notice that the norm of $R(\beta)$ is independent of L and is integrable in zero. To do that, one has to employ the estimates from Proposition 2, (4.38) and the criterion from (4.34).

In order to perform a similar perturbative treatment of the semigroup near a nonzero magnetic field, we need the estimate given by the next proposition:

Proposition 5. *Let \mathbf{n} be a unit vector in \mathbb{R}^3 . Then there exist three positive numerical constants s , c_4 and c_5 such that for all $\omega \in \Omega$, $\mathbf{x}, \mathbf{x}' \in \Lambda$ and $\beta > 0$, one has the following uniform estimate in L :*

$$|\mathbf{n} \cdot (-i\nabla - \omega\mathbf{a}(\mathbf{x})) G_{\omega,L}(\mathbf{x}, \mathbf{x}'; \beta)| \leq c_4 \frac{(1+\beta)^s}{\beta^2} \exp\left[-\frac{|\mathbf{x} - \mathbf{x}'|^2}{c_5\beta}\right]. \quad (4.46)$$

Proof. Proposition 3 allows us to write down the following integral equation:

$$W_L(\beta, \omega) = S(\beta) - \int_0^\beta d\tau W_L(\beta - \tau, \omega) R(\tau). \quad (4.47)$$

Because $S(\beta)$ and W_L are self-adjoint, one can rewrite (4.47) as:

$$W_L(\beta, \omega) = S(\beta) - \int_0^\beta d\tau R^*(\tau) W_L(\beta - \tau, \omega). \quad (4.48)$$

In terms of integral kernels, (4.48) reads as:

$$G_{\omega,L}(\mathbf{x}, \mathbf{x}'; \beta) = S(\mathbf{x}, \mathbf{x}'; \beta) - \int_0^\beta d\tau \int_\Lambda d\mathbf{y} R^*(\mathbf{x}, \mathbf{y}, \tau) G_{\omega,L}(\mathbf{y}, \mathbf{x}', ; \beta - \tau), \quad (4.49)$$

where the equality is between continuous functions in $C(\Lambda \times \Lambda)$ and the integral in τ has to be understood as “ $\int_\epsilon^{\beta-\epsilon}$ ” in the limit $\epsilon \searrow 0$.

Because the kernel of R^* reads as $R^*(\mathbf{x}, \mathbf{y}; \tau) = \overline{R(\mathbf{y}, \mathbf{x}; \tau)}$, by direct computation one can obtain the estimate (see Proposition 2):

$$|\mathbf{n} \cdot (-i\nabla - \omega \mathbf{a}(\mathbf{x})) R^*(\mathbf{x}, \mathbf{x}'; \tau)| \leq c'_4 \frac{(1 + \tau)^{s'}}{\tau^2} \exp \left[-\frac{|\mathbf{x} - \mathbf{x}'|^2}{c'_5 \tau} \right], \quad (4.50)$$

where one has to apply (4.33) then use the estimates from Proposition 2; finally, introducing (4.50), (2.3), (4.35) and (4.38) in (4.49) and because the singularity in τ is integrable, the result for $G_{\omega, L}$ is straightforward.

Take $\omega = \omega_0 + \delta\omega \in \Omega$. The analogous of Proposition 4 at nonzero magnetic field is:

Proposition 6. *The bounded operator denoted with $S_\omega(\beta)$ and given by the kernel $e^{i\delta\omega\varphi(\mathbf{x}, \mathbf{x}')} G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta)$ has the following properties:*

- i. $(0, \infty) \ni \beta \mapsto S_\omega(\beta)$ is strongly differentiable and $s - \lim_{\beta \searrow 0} S_\omega(\beta) = \mathbf{1}$;
- ii. $\text{Ran} S_\omega(\beta) \in \text{Dom}(H_{1, L}(\omega))$ and $\frac{\partial}{\partial \beta} S_\omega(\beta) f + H_{1, L}(\omega) S_\omega(\beta) f = R_\omega(\beta) f$, where $R_\omega(\beta)$ is given by:

$$\begin{aligned} R_\omega(\mathbf{x}, \mathbf{x}'; \beta) &= e^{i(\delta\omega)\varphi(\mathbf{x}, \mathbf{x}')} [(\delta\omega)^2 \mathbf{a}^2(\mathbf{x} - \mathbf{x}') G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta) + \\ &+ 2(\delta\omega) \mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot (i\nabla_{\mathbf{x}} + \omega_0 \mathbf{a}(\mathbf{x})) G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta)]. \end{aligned} \quad (4.51)$$

Proof. i. Rewriting the integral kernel of $S_\omega(\beta)$ as:

$$S_\omega(\mathbf{x}, \mathbf{x}'; \beta) = G_{\omega, L}(\mathbf{x}, \mathbf{x}'; \beta) + \left[e^{i(\delta\omega)\varphi(\mathbf{x}, \mathbf{x}')} - 1 \right] G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta), \quad (4.52)$$

and using the diamagnetic inequality, one can reproduce the argument from (4.40) in order to prove that the second term converges in norm to zero. Clearly, the first one converges strongly to one.

If $\{\psi_j\}$ and $\{E_j\}$ denote the sets of eigenvectors and eigenvalues of $H_{1, L}(\omega)$, then:

$$G_{\omega, L}(\mathbf{x}, \mathbf{x}'; \beta) = \sum_j e^{-\beta E_j} \psi_j(\mathbf{x}) \overline{\psi_j}(\mathbf{x}'), \quad (4.53)$$

where the series is absolutely and uniformly convergent on $\overline{\Lambda} \times \overline{\Lambda}$. This can be seen from the fact that the semigroup is trace class and that the eigenfunctions belong to D_0 and admit the estimate:

$$|\psi_j|(\mathbf{x}) \leq \text{const}(L) (E_j + 1), \quad (4.54)$$

obtained from the fact that the resolvent $[H_{1, L}(\omega) + 1]^{-1}$ is bounded between $\mathbf{L}^2(\Lambda)$ and $\mathbf{L}^\infty(\Lambda)$. It follows that uniformly in Λ :

$$|G_{\omega, L}(\cdot, \cdot; \beta + \delta\beta) - G_{\omega, L}(\cdot, \cdot; \beta) - \frac{\partial G_{\omega, L}}{\partial \beta}(\cdot, \cdot; \beta)| \leq \text{const}(L) (\delta\beta)^2, \quad (4.55)$$

which is sufficient for the strong differentiability (see also (4.41)).

ii. One has to make the same steps as in the proof of the third point of Proposition 4. As for the norm of $R(\beta)$, let us see that is independent of L and

is integrable in zero: from (4.51), (4.46), (4.38) and (2.3), one can obtain an estimate on the kernel of $R_\omega(\beta)$ of the following form:

$$|R_\omega(\mathbf{x}, \mathbf{x}'; \beta)| \leq c_9 \delta\omega (1 + \beta)^s G_{0,\infty}(\mathbf{x}, \mathbf{x}'; c_{10}\beta), \quad (4.56)$$

which implies that its $B(\mathbf{L}^2(\Lambda))$ norm is bounded by a constant multiplied with $\delta\omega$ (see (4.34)).

We shall give now without proof a result which gives sufficient conditions for an operator defined in $B(\mathbf{L}^2(\Lambda))$ to be trace class:

Proposition 7. *Let $\{T_n\}$ a sequence of trace class operators, converging to T in $B(\mathbf{L}^2(\Lambda))$. If $\sup_n \|T_n\|_{\mathcal{I}_1} \leq c < \infty$, then $T \in \mathcal{I}_1$ and $\|T\|_{\mathcal{I}_1} \leq c$.*

Remark. Assume that an operator T is defined by a $B(\mathbf{L}^2(\Lambda))$ -norm Riemann integral on the interval $[a, b]$, with a continuous trace class integrand $S(t)$. If $\int_a^b dt \|S(t)\|_{\mathcal{I}_1} \leq c < \infty$, then T is trace class and

$$\|T\|_{\mathcal{I}_1} \leq c, \quad \text{tr } T = \int_a^b dt \text{tr } S(t).$$

Denote with $\tilde{R}(\omega, \beta)$ the bounded operator given by the kernel:

$$\tilde{R}(\mathbf{x}, \mathbf{x}'; \beta) = 2 \mathbf{a}(\mathbf{x} - \mathbf{x}') \cdot (-i\nabla - \omega_0 \mathbf{a}(\mathbf{x})) G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta). \quad (4.57)$$

Among other things, the next proposition proves (4.10):

Proposition 8. *Take $\beta > 0$ and $\omega = \omega_0 + \delta\omega \in \Omega$.*

i. *The operator $S_\omega(\beta)$ is trace class and moreover, there exists a positive numerical constant c such that:*

$$\|W_L(\beta, \omega) - S_\omega(\beta)\|_{B(\mathbf{L}^2(\Lambda))} \leq c \delta\omega \quad \text{and} \quad (4.58)$$

$$\|W_L(\beta, \omega) - S_\omega(\beta)\|_{\mathcal{I}_1} \leq c \delta\omega L^3. \quad (4.59)$$

ii. *For all $\mathbf{x}, \mathbf{x}' \in \Lambda$ and uniformly in L , one has:*

$$\begin{aligned} G_{\omega, L}(\mathbf{x}, \mathbf{x}'; \beta) &= e^{i\delta\omega\varphi(\mathbf{x}, \mathbf{x}')} G_{\omega_0, L}(\mathbf{x}, \mathbf{x}'; \beta) + \\ &+ \delta\omega \int_0^\beta d\tau \int_\Lambda d\mathbf{y} e^{i\delta\omega\varphi(\mathbf{x}, \mathbf{y})} G_{\omega_0, L}(\mathbf{x}, \mathbf{y}; \beta - \tau) e^{i\delta\omega\varphi(\mathbf{y}, \mathbf{x}')} \tilde{R}(\mathbf{y}, \mathbf{x}'; \tau) + \mathcal{O}((\delta\omega)^2). \end{aligned} \quad (4.60)$$

Proof. i. We know that as bounded operators:

$$W_L(\beta, \omega) = S_\omega(\beta) - \int_0^\beta d\tau W_L(\beta - \tau, \omega) R_\omega(\tau). \quad (4.61)$$

We have already seen that the $B(\mathbf{L}^2(\Lambda))$ norm of $R_\omega(\tau)$ is bounded by a constant multiplied with $\delta\omega$ (see (4.56)). Its Hilbert-Schmidt norm is bounded by (see (4.37)):

$$\|R_\omega(\tau)\|_{\mathcal{I}_2} \leq \text{const } \delta\omega \frac{(1 + \tau)^s}{\tau^{3/4}} L^{3/2}. \quad (4.62)$$

For the semigroup, we know that $\|W_L\|_{B(\mathbf{L}^2(\Lambda))} \leq 1$ and from (2.3) and (4.37) it follows that:

$$\|W_L(\beta - \tau, \omega)\|_{\mathcal{I}_2} \leq \text{const} \frac{1}{(\beta - \tau)^{3/4}} L^{3/2}. \quad (4.63)$$

With the help of the well known inequality $\|A B\|_{\mathcal{I}_1} \leq \|A\|_{\mathcal{I}_2} \|B\|_{\mathcal{I}_2}$, it follows that in both situations ($B(\mathbf{L}^2)$ and $\mathcal{I}_1(\mathbf{L}^2)$) the singularities in τ are integrable (see the previous remark), and the desired bounds follow easily.

ii. The formula (4.60) is obtained from (4.61) by isolating the term which contains $\delta\omega$.

The next proposition imposes sufficient conditions on a trace class integral operator such that its trace to be equal to the integral of the kernel's diagonal (see [R-S 1]):

Proposition 9. *Let $T \in \mathcal{I}_1(\mathbf{L}^2(\Lambda))$, given by the integral kernel $T(\mathbf{x}, \mathbf{x}') \in C(\overline{\Lambda} \times \overline{\Lambda})$. Then:*

$$\text{tr } T = \int_{\Lambda} d\mathbf{x} T(\mathbf{x}, \mathbf{x}). \quad (4.64)$$

The rest of this subsection is dedicated to the proof of (4.9). Fix $\beta, \tau > 0$. Let $0 < \omega_0 < \omega_1$ and let $\Omega = [\omega_0, \omega_1]$. Clearly, $g_\omega(\xi, z; \beta, \tau)$ is trace class and admits a continuous integral kernel given by the following series, which is absolutely and uniformly convergent on $\Lambda \times \Lambda$ (see also (4.53)):

$$T_\omega(\mathbf{x}, \mathbf{x}') = \sum_j [\xi - z \exp(-\beta E_j)]^{-1} z \exp(-\tau E_j) \psi_j(\mathbf{x}) \overline{\psi_j}(\mathbf{x}'), \quad (4.65)$$

Notice that in order to simplify the notations, we did not specify the dependence on ξ, z, β and τ .

Let us start with the equation satisfied by $T_\omega(\mathbf{x}, \mathbf{x}')$:

Proposition 10. *As continuous functions:*

$$\xi T_\omega(\mathbf{x}, \mathbf{x}') - z \int_{\Lambda} d\mathbf{y} G_{\omega, L}(\mathbf{x}, \mathbf{y}; \beta) T_\omega(\mathbf{y}, \mathbf{x}') = z G_{\omega, L}(\mathbf{x}, \mathbf{x}'; \tau). \quad (4.66)$$

Proof. The above equality is nothing but the rewriting in terms of integral kernels of an identity between bounded operators:

$$[\xi - z W_L(\beta, \omega)] g_\omega(\xi, z; \beta, \tau) = z W_L(\tau, \omega). \quad (4.67)$$

For further purposes, we shall prove that for all $L \geq 1$, $|T_\omega(\mathbf{x}, \mathbf{x}')| \sim e^{-\alpha|\mathbf{x}-\mathbf{x}'|}$ for some positive α . We need first a few definitions: let $\rho(\mathbf{x}) := (1 + \mathbf{x}^2)^{1/2}$ and $\alpha \geq 0$. It is known that the partial derivatives up to the second order of ρ are bounded by a numerical constant and moreover:

$$\forall \mathbf{x} \in \mathbb{R}^3, e^{\pm \alpha \rho(\mathbf{x})} e^{\mp \alpha |\mathbf{x}|} \leq \text{const}(\alpha). \quad (4.68)$$

Fix $\mathbf{x}_0 \in \Lambda$. Denote with $A(\alpha)$ the multiplication operator with $e^{-\alpha \rho(\cdot - \mathbf{x}_0)}$. Then $A(\alpha)$ and $A^{-1}(\alpha) = A(-\alpha)$ are bounded operators and invariate D_0 (see 4.42)). An useful result is contained in the next proposition:

Proposition 11. i. The operator $A(\alpha)W_L(\tau, \omega)A(-\alpha)$ belongs to $B(\mathbf{L}^2)$, and has a norm which is uniformly bounded in L , $\mathbf{x}_0 \in \Lambda$, $\omega \in \Omega$ and $0 < \tau \leq \beta$;

ii. The operator $A(\alpha)W_L(\beta, \omega)A(-\alpha)$ belongs to $B(\mathbf{L}^2, \mathbf{L}^\infty)$, having a norm which is uniformly bounded in L , ω and \mathbf{x}_0 .

Proof. i. The inequality (4.68) allows us to replace A with the multiplication operator given by $e^{\alpha|\mathbf{x}-\mathbf{x}_0|}$. Let $\psi \in \mathbf{L}^2(\Lambda)$ and define:

$$\phi := A(\alpha)W_L(\tau, \omega)A(-\alpha) = \int_{\Lambda} d\mathbf{y} e^{-\alpha\rho(\cdot-\mathbf{x}_0)} G_{L, \omega}(\cdot, \mathbf{y}; \tau) e^{\alpha\rho(\mathbf{y}-\mathbf{x}_0)} \psi(\mathbf{y}). \quad (4.69)$$

Let us remark an elementary estimate, which is true for all $0 < \tau \leq \beta$:

$$e^{\alpha|\mathbf{x}-\mathbf{x}'|} \exp \left[-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\tau} \right] \leq \text{const}(\alpha, \beta). \quad (4.70)$$

Applying (4.68), the triangle inequality, (2.3) and (4.70) one obtains:

$$\begin{aligned} |\phi|(\mathbf{x}) &\leq \text{const}(\alpha) \int_{\Lambda} d\mathbf{x}' e^{\alpha|\mathbf{x}-\mathbf{x}'|} |G_{\omega, L}(\mathbf{x}, \mathbf{x}'; \tau)| |\psi|(\mathbf{x}') \leq \\ &\leq \text{const}(\alpha, \beta) \int_{\Lambda} d\mathbf{x}' G_{0, \infty}(\mathbf{x}, \mathbf{x}'; 2\tau) |\psi|(\mathbf{x}'). \end{aligned} \quad (4.71)$$

The result follows from (4.36) and (4.34).

ii. We apply the Schwartz inequality in (4.71) with $\tau = \beta$ and then use (4.37).

Proposition 12. Under the same conditions as above, there exists a sufficiently small $0 < \alpha < 1$ such that the following inequalities are true in $B(\mathbf{L}^2(\Lambda))$, uniformly in L , ω and \mathbf{x}_0 :

- i. $\|W_L(\beta, \omega) - A(\alpha)W_L(\beta, \omega)A(-\alpha)\| \leq \alpha \text{const}(\beta)$;
- ii. Uniformly in $\xi \in \mathcal{C}$ and $z \in K$ one has:

$$\|A(\alpha)[\xi - zW_L(\beta, \omega)]^{-1}A(-\alpha)\|_{B(\mathbf{L}^2(\Lambda))} \leq \text{const}(\beta).$$

Proof. i. Let $S(\beta) = A(\alpha)W_L(\beta, \omega)A(-\alpha)$. We will see that $S(\beta)$ obeys the conditions of Proposition 3. From Proposition 11 follows condition A. Then $S(\beta)$ is strongly differentiable, has its range included in the domain of $H_{1, L}(\omega)$ and converges strongly to one. Define $B := H_{1, L}(\omega) - A(\alpha)H_{1, L}(\omega)A(-\alpha)$, or in other form:

$$B = 2i\alpha(\mathbf{p} - \omega\mathbf{a}) \cdot \nabla \rho(\cdot - \mathbf{x}_0) + \alpha(\Delta \rho)(\cdot - \mathbf{x}_0) - \alpha^2 |\nabla \rho(\cdot - \mathbf{x}_0)|^2. \quad (4.72)$$

A well known result says (see [S]) that $\|B[H_{1, L}(\omega) + 1]^{-1/2}\| \leq \text{const}$.

By direct computation, $R(\tau) = B A(\alpha)W_L(\tau, \omega)A(-\alpha)$; a rough estimate gives $\|R(\tau)\| \leq \text{const}(L)/\sqrt{\tau}$ and even if the constant behaves badly with L , the norm is integrable in zero with respect to τ .

In conclusion:

$$\begin{aligned} W_L(\beta, \omega) - S(\beta) &= - \int_0^\beta d\tau W_L(\beta - \tau, \omega) R(\tau) \\ &= - \int_0^\beta d\tau W_L(\beta - \tau, \omega) B S(\tau), \end{aligned} \quad (4.73)$$

where the integral converges in norm. But uniformly in L and \mathbf{x}_0 there exists a numerical constant such that:

$$\|W_L(\beta - \tau, \omega)B\| \leq \alpha \frac{\text{const}}{\sqrt{\beta - \tau}}. \quad (4.74)$$

From (4.74), (4.73) and point i of Proposition 11, the needed estimate follows.

ii. Using point i, the estimate (4.3) and the identity (α small enough):

$$\begin{aligned} A(\alpha)[\xi - zW_L(\beta, \omega)]^{-1}A(-\alpha) &= [\xi - zA(\alpha)W_L(\beta, \omega)A(-\alpha)]^{-1} = \\ &= \sum_{j \geq 0} [\xi - zW_L(\beta, \omega)]^{-1} z^j \cdot \\ &\cdot \{ [A(\alpha)W_L(\beta, \omega)A(-\alpha) - W_L(\beta, \omega)][\xi - zW_L(\beta, \omega)]^{-1} \}^j, \end{aligned}$$

the result follows.

Corollary 1. *The operator $A(\alpha)g_\omega(\xi, z; \beta, \tau)A(-\alpha)$ belongs to $B(\mathbf{L}^2, \mathbf{L}^\infty)$ if α is small enough, and uniformly in $\xi, z, \mathbf{x}_0, \omega$ and L one has:*

- i. $\|A(\alpha)g_\omega(\xi, z; \beta, \tau)A(-\alpha)\| \leq \text{const}(\beta, \tau);$
- ii. $\int_A d\mathbf{y} e^{2\alpha|\mathbf{y}-\mathbf{x}_0|} |T_\omega(\mathbf{x}_0, \mathbf{y})|^2 \leq \text{const}(\beta, \tau);$
- iii. $e^{\alpha|\mathbf{x}-\mathbf{y}|} |T_\omega(\mathbf{x}, \mathbf{y})| \leq \text{const}(\beta, \tau).$

Proof. i. It is an immediate consequence of Propositions 11 and 12.

ii. Let $\phi = A(\alpha)g_\omega(\xi, z; \beta, \tau)A(-\alpha)\psi$, where ϕ is bounded and continuous. From i it follows:

$$|\phi(\mathbf{x}_0)| = \left| \int_A d\mathbf{y} e^{\alpha\rho(\mathbf{y}-\mathbf{x}_0)} T_\omega(\mathbf{x}_0, \mathbf{y}) \psi(\mathbf{y}) \right| \leq \text{const}(\beta, \tau) \|\psi\|_{\mathbf{L}^2}, \quad (4.75)$$

and the result follows from the representation theorem of linear and continuous functionals on \mathbf{L}^2 .

iii. Rewrite the identity $g_\omega(\xi, z; \beta, \tau) = g_\omega(\xi, z; \beta, \tau/2)W_L(\tau/2, \omega)$ in terms of integral kernels, use (4.68), (2.3), ii, the triangle and Schwartz inequalities, and the proof is completed.

Let $\delta\omega > 0$ be such that $\omega = \omega_0 + \delta\omega \in \Omega$. Define the bounded operator $\tilde{g}_\omega(\xi, z; \beta, \tau)$ given by the following integral kernel:

$$\tilde{T}_\omega(\mathbf{x}, \mathbf{x}') := e^{i\delta\omega\varphi(\mathbf{x}, \mathbf{x}')} T_{\omega_0}(\mathbf{x}, \mathbf{x}'). \quad (4.76)$$

Equations (4.3) and (4.58) imply that if $\delta\omega$ is sufficiently small then there exists a numerical constant such that uniformly in L :

$$\sup_{\xi} \sup_z \|[\xi - zS_\omega(\beta)]^{-1}\| \leq \text{const} \quad (4.77)$$

and:

$$\sup_{\xi} \sup_z \|[\xi - zS_\omega(\beta)]^{-1} - [\xi - zW_L(\beta, \omega)]^{-1}\| \leq \text{const } \delta\omega. \quad (4.78)$$

We state now an important property of $\tilde{g}_\omega(\xi, z; \beta, \tau)$:

Proposition 13. *Under the above conditions, there exists a numerical constant such that if $\delta\omega$ is small enough, then uniformly in ξ , z and L , the following $B(\mathbf{L}^2)$ estimate takes place:*

$$||[\xi - zS_\omega(\beta)]^{-1}zS_\omega(\tau) - \tilde{g}_\omega(\xi, z; \beta, \tau)|| \leq \text{const } \delta\omega. \quad (4.79)$$

Proof. The integral kernel of the operator $[\xi - zS_\omega(\beta)]\tilde{g}_\omega(\xi, z; \beta, \tau)$ is given by:

$$\xi\tilde{T}_\omega(\mathbf{x}, \mathbf{x}') - z \int_A d\mathbf{y} S_\omega(\mathbf{x}, \mathbf{y}; \beta)\tilde{T}_\omega(\mathbf{y}, \mathbf{x}'). \quad (4.80)$$

Let us notice a crucial property of the magnetic phase:

$$\varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{y}, \mathbf{x}') = \varphi(\mathbf{x}, \mathbf{x}') + \text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}'), \quad (4.81)$$

where $\text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}') = 1/2 \mathbf{B} \cdot [(\mathbf{y} - \mathbf{x}') \wedge (\mathbf{x} - \mathbf{y})]$. Using (4.81) and (4.66) in (4.80) we obtain:

$$[\xi - zS_\omega(\beta)]\tilde{g}_\omega(\xi, z; \beta, \tau) = zS_\omega(\tau) + R, \quad (4.82)$$

where R is an integral operator given by:

$$-z e^{i\delta\omega\varphi(\mathbf{x}, \mathbf{x}')} \int_A d\mathbf{y} (e^{i\delta\omega \text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1) G_{\omega_0, L}(\mathbf{x}, \mathbf{y}; \beta) T_{\omega_0}(\mathbf{y}, \mathbf{x}'). \quad (4.83)$$

Because $|e^{i\delta\omega \text{fl}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1| \leq \delta\omega |\mathbf{x} - \mathbf{y}| |\mathbf{y} - \mathbf{x}'|$, denoting with P the operator given by $|\mathbf{x} - \mathbf{y}| |zG_{\omega_0, L}(\mathbf{x}, \mathbf{y}; \beta)|$ and with Q the operator corresponding to $|\mathbf{y} - \mathbf{x}'| |T_{\omega_0}(\mathbf{y}, \mathbf{x}')|$, it follows that $\|R\| \leq \delta\omega \|P\| \|Q\|$. Using (2.3), Corollary 1 iii., (4.34) and (4.77), the proof is completed.

Employing (4.58), (4.79), (4.78) and (4.77) in the next equality:

$$\begin{aligned} g_\omega(\xi, z; \beta, \beta/2) - \tilde{g}_\omega(\xi, z; \beta, \beta/2) &= \\ &= [(\xi - zW_L(\beta, \omega))^{-1} - (\xi - zS_\omega(\beta))^{-1}]zW_L(\beta/2, \omega) + \\ &+ (\xi - zS_\omega(\beta))^{-1}z[W_L(\beta/2, \omega) - S_\omega(\beta/2)] + \\ &+ (\xi - zS_\omega(\beta))^{-1}zS_\omega(\beta/2) - \tilde{g}_\omega(\xi, z; \beta, \beta/2), \end{aligned} \quad (4.84)$$

(4.9) is straightforward.

Let us end this subsection by proving that the operator

$$A_\omega(\xi, z; \beta) = \tilde{g}_\omega(\xi, z; \beta, \beta/2)S_\omega(\beta/2)$$

has the same trace as $g_{\omega_0}(\xi, z; \beta, \beta)$. Indeed, because A_ω fulfills the conditions of Proposition 9 and noticing that $\varphi(\mathbf{x}, \mathbf{x}') = -\varphi(\mathbf{x}', \mathbf{x})$, one can write:

$$\begin{aligned} \text{tr} A_\omega(\xi, z; \beta) &= \int_{\Lambda^2} d\mathbf{x} d\mathbf{x}' T_{\omega_0}(\xi, z; \beta, \beta/2; \mathbf{x}, \mathbf{x}') G_{\omega_0, L}(\mathbf{x}', \mathbf{x}; \beta/2) \\ &= \int_A d\mathbf{x} T_{\omega_0}(\xi, z; \beta, \beta; \mathbf{x}, \mathbf{x}) = \text{tr} g_{\omega_0}(\xi, z; \beta, \beta). \end{aligned} \quad (4.85)$$

4.2. *The proof of II and III.* The analyticity of $\Gamma_\infty(\beta, z, \omega_0)$ in \mathcal{D} follows from the bound (see (2.16)) $|g_\sigma(\zeta)| \leq \text{const}(\sigma, K)|\zeta|$ where K is some compact in $\mathbb{C} \setminus [1, \infty)$.

In what follows, we will prove that if $z \in \mathcal{D}_0 := \{|z| < 1\}$, then:

$$\lim_{L \rightarrow \infty} \Gamma_L(\beta, z, \omega_0) = \Gamma_\infty(\beta, z, \omega_0). \quad (4.86)$$

Because $|z| < 1$, the grand canonical pressure will be (see (2.10)):

$$P_L(\beta, z, \omega) = \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1}{\beta L^3} \text{tr} W_L(n\beta, \omega) = \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\frac{1}{\beta L^3} \int_{\Lambda} d\mathbf{x} G_{\omega, L}(\mathbf{x}, \mathbf{x}; n\beta) \right). \quad (4.87)$$

Under the same conditions, the magnetization reads as:

$$\Gamma_L(\beta, z, \omega_0) = -\frac{e}{c} \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1}{\beta L^3} \text{tr} \left[\frac{\partial W_L}{\partial \omega} \right] (n\beta, \omega_0). \quad (4.88)$$

An important quantity is the integral kernel of the semigroup defined on the whole space:

$$G_{\omega, \infty}(\mathbf{x}, \mathbf{x}'; \beta) = \frac{e^{i\omega\varphi(\mathbf{x}, \mathbf{x}')}}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)}. \quad (4.89)$$

$$\cdot \exp \left\{ -\frac{1}{2\beta} \left[\frac{\omega\beta/2}{\tanh(\omega\beta/2)} (\mathbf{e}_3 \wedge (\mathbf{x} - \mathbf{x}'))^2 + (\mathbf{e}_3 \cdot (\mathbf{x} - \mathbf{x}'))^2 \right] \right\}. \quad (4.90)$$

Denote with:

$$g(\beta, \omega) = G_{\omega, \infty}(\mathbf{x}, \mathbf{x}; \beta) = \frac{1}{(2\pi\beta)^{3/2}} \frac{\omega\beta/2}{\sinh(\omega\beta/2)}.$$

Then it is easy to see that if $|z| < 1$:

$$P_\infty(\beta, z, \omega_0) = \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{g(n\beta, \omega_0)}{\beta}, \quad \Gamma_\infty(\beta, z, \omega_0) = -\frac{e}{\beta c} \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{\partial g}{\partial \omega}(n\beta, \omega_0).$$

One of the results in [M-M-P 1] can be adapted to our problem and gives:

$$\lim_{L \rightarrow \infty} \frac{1}{L^3} \text{tr} \left[\frac{\partial W_L}{\partial \omega} \right] (\beta, \omega_0) = \frac{\partial g}{\partial \omega}(\beta, \omega_0). \quad (4.91)$$

If we prove the existence of a positive function f with at most polynomial growth such that:

$$\left| \frac{1}{L^3} \text{tr} \left[\frac{\partial W_L}{\partial \omega} \right] \right| (n\beta, \omega_0) \leq f(n\beta), \quad (4.92)$$

then (4.86) would be true.

The next corollary is a direct consequence of Proposition 8 ii.:

Corollary 2. *Under the conditions of Proposition 8 one has:*

$$\begin{aligned} & \lim_{\delta\omega \searrow 0} \frac{1}{\delta\omega} \int_A d\mathbf{x} [G_{\omega_0+\delta\omega,L}(\mathbf{x}, \mathbf{x}; \beta) - G_{\omega_0,L}(\mathbf{x}, \mathbf{x}; \beta)] = \\ & = \text{tr} \left[\frac{\partial W_L}{\partial \omega} \right] (\beta, \omega_0) = 2 \int_0^\beta d\tau \int_{\Lambda^2} d\mathbf{x} d\mathbf{y} \cdot \\ & \cdot G_{\omega_0,L}(\mathbf{x}, \mathbf{y}; \beta - \tau) \mathbf{a}(\mathbf{y} - \mathbf{x}) \cdot (-i\nabla_{\mathbf{y}} - \omega_0 \mathbf{a}(\mathbf{y})) G_{\omega_0,L}(\mathbf{y}, \mathbf{x}; \tau). \end{aligned} \quad (4.93)$$

Use (2.3), (4.46), (4.38) and (4.35) in (4.93) and (4.92) follows. The proof of (4.86) is now completed.

Remarks. 1. What can we say about the same problem for Fermi particles (say electrons, where $H_{1,L}(\omega)$ should be replaced with the Pauli operator)? Knowing that in this case $\inf \sigma(H_{1,\infty}(\omega)) = 0$ (i.e. is independent of ω), the grand canonical result (Lemma 1) can be easily restated in terms of Fermi statistics: the only thing that changes is the domain on which the limit takes place i.e. $\mathbb{C} \setminus (-\infty, -1]$. As for Theorem 2, its proof was based on the fact that there exists a compact $K \subset \mathbb{C} \setminus [e^{\beta\omega/2}, \infty)$ which contains the circle centered in the origin with radius equal to $x_L(\beta, \rho, \omega)$, for all $L \geq L_0$ and for all strictly positive β , ω and ρ . For Fermi particles, if one fixes ρ and ω but makes β very large (lowers the temperature), then $x_L(\beta, \rho, \omega)$ would be a very large positive quantity, therefore the above circle could intersect the negative cut. Of course, if the gas is diluted (say ρ and ω fixed and β small), then it could happen that $x_\infty(\beta, \rho, \omega) < 1$ which means that the circle never intersects the cut when $L \geq L_0$, therefore a similar proof can be provided. Our conclusion is that the extension of Theorem 2 to a Fermi gas at low temperature is not trivial and remains an interesting problem.

2. What about the higher derivatives with respect to ω (the susceptibility for example) at $\omega_0 \neq 0$? This also remains an open problem, even for the grand canonical ensemble. Nevertheless, we think that our approach (the modified perturbation theory for Gibbs semigroups) could provide an answer to it.

Acknowledgement. Part of this work was done during a visit to Centre de Physique Théorique in Marseille, at the invitation of Professor P. Duclos. The financial support of CNCSU grant 13(C) is hereby gratefully acknowledged. Finally, the author wishes to thank Professors N. Anghel, M. Bundaru and G. Nenciu for their encouragement and fruitful discussions.

References

- A. Anghel, N.: Ph.D thesis, I.F.A., Bucharest, 1976
- A-B-N 1. Anghel, N., Bundaru, M., Nenciu, G.: On the perturbation of Gibbs semigroups. Commun. Math. Phys. **42**, 29-30 (1975)
- A-B-N 2. Anghel, N., Bundaru, M., Nenciu, G.: On the Landau diamagnetism. Commun. Math. Phys. **42**, 9-28 (1975)
- A-C. Anghel, N., Corciovei, A.: On free quantum gases in a homogeneous magnetic field. Rev. Roum. Phys. **20**, 661-671 (1975)
- B-H-L. Broderix, K., Hundertmark, D., Leschke, H.: Continuity properties of Schrodinger semigroups with magnetic fields. Mathematical-Physics preprint archive of University of Texas at Austin.

- C-N. Cornean, H.D., Nenciu, G.: On eigenfunction decay for two dimensional magnetic Schrödinger operators. *Commun. Math. Phys.* **192**, 671-685 (1998)
- H. Huang, K.: Statistical mechanics. New York-London: John Wiley & Sons, Inc., 1963
- H-P. Hille, E., Phillips, R.S.: Functional integral and semigroups. Providence; RI: Am. Math. Soc., 1957
- K-U-Z. Kac, M., Uhlenbeck, G.E., Ziff, R.M.: The ideal Bose-Einstein gas, revisited. *Phys. Rep.* **32C**, *n°* 4, 169-248 (1977)
- K. Kunz, H.: Surface orbital magnetism. *J. Stat. Phys.* **76**, 183-207 (1994)
- M-M-P 1. Macris, N., Martin, Ph.A., Pulé, J.V.: Diamagnetic currents. *Commun. Math. Phys.* **117**, 215-241 (1988)
- M-M-P 2. Macris, N., Martin, Ph.A., Pulé, J.V.: Large volume asymptotics of Brownian integrals and orbital magnetism. *Ann. I.H.P. Phys. Theor.* **66**, 147-183 (1997)
- R-S 1,4. Reed, M., Simon, B.: Methods of modern mathematical physics **I**, **IV**. New York: Academic Press, 1975
- S. Simon, B.: Schrödinger semigroups. *Bull. Am. Math. Soc. (N. S.)* **7**, 447-510 (1982)

Communicated by H. Araki